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ON LEVEQUE'S THEOREM ABOUT THE UNIFORM DISTRIBUTION (MOD 1) OF $(a_j \cos a_j x)_{j=1}^{\infty}$

BY

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ABSTRACT

We prove some refinements of the theorem mentioned in the title.

1. Introduction

Throughout this paper $(a_j)_{j=1}^{\infty}$ denotes a sequence of strictly increasing integers. It was shown by W. J. LeVeque [12] that for almost all x with respect to Lebesgue measure, the sequence $(\langle a_j \cos a_j x\rangle)_{j=1}^{\infty}$ is uniformly distributed modulo one. Here for any real number y, $\langle y\rangle = y - [y]$, where $[y]$ denotes the largest integer not greater than y. In this paper, adapting ideas used by R. C. Baker in [1] and [2] to study sequences like $(\langle a_j x\rangle)_{j=1}^{\infty}$, we prove some refinements of LeVeque's theorem. Except in Theorem 1 this is done under additional assumptions about the rate of growth of the sequence $(a_j)_{j=1}^{\infty}$. Before we are in a position to give more details however, we need to introduce a piece of terminology.

Definition. By the discrepancy of $x_1, \ldots, x_n \in [0, 1)$ we mean

$$D(x_1, \ldots, x_n) = \sup \left| \frac{1}{n} \sum_{j=1}^{n} \chi_I(x_j) - |I| \right| .$$

Here for any subset B of $[0, 1)$, by $\chi_B(x)$ we mean its characteristic function and

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if it is Lebesgue measurable, by $|B|$ its Lebesgue measure. The above supremum is taken over all intervals I contained in $[0, 1)$, which are open on the right and closed on the left.

As is well known a sequence of real numbers $(y_j)_{j=1}^{\infty}$ is uniformly distributed modulo one if and only if

$$D(\langle y_1 \rangle, \ldots, \langle y_n \rangle) = o(1)$$

(see [10], p. 89). To be brief, for each pair of integers $m \geqq 0$ and $n \geqq 1$, let

$$D(m, n, x) = D(\langle a_{m+1} \cos a_{m+1} x \rangle, \ldots, \langle a_{m+n} \cos a_{m+n} x \rangle)$$

and let

$$D(n, x) = D(0, n, x).$$

Put in the language of discrepancy LeVeque's theorem becomes

$$D(n, x) = o(1) \quad \text{a.e.}$$

Here and henceforth we adopt the convention that we mention the measure we are dealing with explicitly only if it is not Lebesgue measure.

In Section 2 the following theorem is proved:

THEOREM 1. *For all* $\varepsilon > 0$, $D(n, x) = o(n^{-1/4}(\log n)^{3/2+\varepsilon})$ *a.e.*

The basic method of proof of this theorem is the same as that used by P. Erdös and J. F. Koksma [4] to prove a result, which as a special case shows that, given $\varepsilon > 0$,

$$D(\langle a_1 x \rangle, \ldots, \langle a_n x \rangle) = o(n^{-1/2}(\log n)^{5/2+\varepsilon}) \quad \text{a.e.} \tag{1}$$

Subsequent to the result (1), which was also proved independently by J. W. S. Cassels [3], attention turned to the exceptional sets themselves. It was shown, by P. Erdös and S. J. Taylor [5], assuming

$$a_j = O(j^p) \qquad \text{for some } p \geqq 1, \tag{2}$$

that if

$$E = \left\{ x : \varlimsup_{n \to \infty} D(\langle a_1 x \rangle, \ldots, \langle a_n x \rangle) > 0 \right\}$$

and if, here and henceforth, dim M denotes the Hausdorff dimension of M, then

$$\dim E \leqq 1 - \frac{1}{p}.$$

Later refinements were added by R. C. Baker who in [2] studied the set

$$E_q = \left\{ x : \overline{\lim_{n \to \infty}}\, n^q D(\langle a_1 x \rangle, \ldots, \langle a_n x \rangle) > 0 \right\}.$$

Assuming in addition to (2) that $0 < q < \frac{1}{2}$, he showed that

$$\dim E_q \leqq 1 - \frac{1 - 2q}{p + q}.$$

In Section 3 we adapt these methods to prove Theorem 2.

Theorem 2. *Suppose $q \in (0, \frac{1}{4})$, for some $p \geqq 1$ that $a_j = O(j^p)$ and that*

$$E_q = \left\{ x : \overline{\lim_{n \to \infty}}\, n^q D(n, x) > 0 \right\}.$$

Then

$$\dim E_q \leqq 1 - \frac{1 - 4q}{4p + 2q + \frac{1}{2}}.$$

In Section 4 we assume for a_j $(j = 1, 2, \ldots)$ that there exist positive constants C_1 and C_2 such that $C_1 j^p \leqq a_j \leqq C_2 j^p$, for some $p \geqq 1$. Here we consider subsets B of $[0, 1)$ with the property that

$$B = \bigcup_{m=1}^{\infty} I_m$$

for a disjoint collection of intervals $(I_m)_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} \frac{\log(|I_m|^{-1})}{\log m} = b > 3\tfrac{4}{7}. \tag{3}$$

Following Theorem 2 of [1], we are interested in the set

$$E(B) = \left\{ x \in [0, 2\pi) : \overline{\lim_{n \to \infty}}\, \frac{1}{n} |F(B, n, x)| > 0 \right\}$$

where

$$F(B, n, x) = \sum_{j=1}^{n} \chi_B(\langle a_j \cos a_j x \rangle) - n|B|.$$

In this event consider the following two polynomials:

$$f_1(y) = \left(1 - p(1-y) - \left(\frac{7-y}{8}\right)\right)\left(\frac{by}{2} - 1\right) - p\left(1 - \frac{3y}{2}\right)\left(\frac{3-y}{2}\right) \tag{4}$$

and

$$f_2(y) = \left(1 - p(1-y) - \left(\frac{7-y}{8}\right)\right)(by-1) - p(1-y)(3-y). \tag{5}$$

The coefficients of y^2 in both $f_1(y)$ and $f_2(y)$ are positive so they each have one simple root after any negative value of the function. This means that as $f_1(2/b) < 0$ and $f_1(1) > 0$, $f_1(y)$ has a root γ_1 (say) in $(2/b, 1)$ and as

$$f_2\left(1 - \frac{1}{4p + \frac{1}{2}}\right) < 0 \quad \text{and} \quad f_2(1) > 0,$$

$f_2(y)$ has a root γ_2 (say) in $(1 - 1/(4p + \frac{1}{2}), 1)$. Let $\gamma = \max(\gamma_1, \gamma_2)$, then we have the following theorem.

THEOREM 3. $\dim E(B) \leqq \gamma$.

2. Lebesgue measure and discrepancy estimates

Before we proceed with the proof of Theorem 1 we need some lemmas. The first of these is a version of a theorem due to I. S. Gál and J. F. Koksma [8].

LEMMA 4. *Let, for each pair of non-negative integers $m \geqq 0$ and $n \geqq 1$,*

$$F(m, n) = F(m, n, x)$$

denote a positive Borel measurable function of x on $[a, b]$. Suppose that whenever $0 \leqq l \leqq n$ we have

$$|F(m, n)| \leqq |F(m, l)| + |F(m + l, n - l)|. \tag{6}$$

Suppose further for some $\theta > 1$, $\phi \geqq 0$ and $\psi \geqq 0$ that

$$\int_a^b |F(m, n, x)|^2 d\mu(x) = O(n^\theta (n + m)^\phi (\log n)^\psi), \tag{7}$$

where μ is a positive finite Borel measure on $[a, b]$. Then for every $\varepsilon > 0$

$$F(0, n, x) = o(n^{(\theta+\phi)/2} (\log n)^{(\psi+1)/2+\varepsilon}) \qquad \mu \ a.e.$$

The original Gál–Koksma theorem was proved with μ being Lebesgue measure. The proof goes through, however, without change for an arbitrary positive finite Borel measure. If we apply Lemma 1 to $F(m, n, x) = nD(m, n, x)$, which clearly satisfies (6), we reduce the proof of Theorem 1 to obtaining estimates of the type (7) for $nD(m, n, x)$ $(n = 1, 2, \ldots, m = 1, 2, \ldots)$. The task is further reduced by Minkowski's inequality and Lemma 5 (which follows and is the well known Erdös–Turán theorem [6]) to estimating the L^2-norms of certain exponential sums. Henceforth in this paper, for a real number x, $e(x)$ will denote $e^{2\pi i x}$.

Lemma 5. *There exists an absolute constant $K_1 > 0$ such that for all $x_1, \ldots, x_n \in [0, 1)$ and any positive integer r*

$$nD(x_1, \ldots, x_n) \leqq K_1 \left(\frac{n}{r} + \sum_{h=1}^{r} \frac{1}{h} \left| \sum_{j=1}^{n} e(hx_j) \right| \right).$$

To obtain the desired L^2-estimate of the exponential sums, we need a "quasi-orthogonality" inequality essentially due to LeVeque [12] which we formulate as Lemma 6.

Lemma 6. *For all positive integers h, j and k $(j \neq k)$ and given interval $[u, v]$, there exists a positive constant $K_2 = K_2(u, v)$, such that*

$$\left| \int_u^v e(h(a_j \cos a_j x - a_k \cos a_k x)) dx \right| \leqq \frac{K_2}{|a_j - a_k|^{1/2}}.$$

Our next lemma completes our estimate of the L^2-norm of the exponential sums.

Lemma 7. *For non-negative integers h, m and $n \geqq 1$ let*

$$S_h(m, n, x) = \sum_{j=m+1}^{m+n} e(ha_j \cos a_j x).$$

There exists a positive constant $K_3 = K_3(u, v)$ such that

$$\left(\int_u^v |S_h(m, n, x)|^2 dx \right)^{1/2} \leqq K_3 n^{3/4}.$$

Proof.

$$\int_u^v |S_h(m, n, x)|^2 dx = \int_u^v \left(\sum_{j=m+1}^{m+n} \sum_{k=m+1}^{m+n} e(h(a_j \cos a_j x - a_k \cos a_k x)) \right) dx,$$

which is

$$\ll \left(n + \sum_{j \neq k} \left| \int_u^v e(h(a_j \cos a_j x - a_k \cos a_k x)) dx \right| \right).$$

Pairing symmetric terms in the above sum and using Lemma 6 this is

$$\ll \left(n + \sum_{m+1 \leqq j < k \leqq m+n} |a_j - a_k|^{-1/2} \right),$$

which, using the fact that $|j - k| \leqq |a_j - a_k|$, is

$$\ll \left(n + \sum_{m+1 < j < k \leqq m+n} (k - j)^{-1/2} \right) \ll n^{3/2},$$

as required. □

We are now in a position to complete the *proof of Theorem 1*. Firstly, Minkowski's inequality and Lemma 7 give us for all positive integers $m, n \geqq 1$ and r

$$\left(\int_a^b (nD(m, n, x))^2 dx \right)^{1/2} \ll \left(\frac{n}{r} + \sum_{h=1}^{r} \frac{1}{h} \left(\int_a^b |S_h(m, n, x)|^2 dx \right)^{1/2} \right).$$

Choosing $r = [n^{1/4}]$, $u = a$ and $v = b$, and quoting Lemma 7,

$$\left(\int_a^b (nD(m, n, x))^2 dx \right)^{1/2} \ll n^{3/4} (\log n).$$

Lemma 4 now gives, for all $\varepsilon > 0$,

$$D(n, x) = o(n^{-1/4} (\log n)^{3/2 + \varepsilon}) \quad \text{a.e.},$$

as required. □

3. Hausdorff dimension of exceptional sets: Discrepancy

Throughout this section we assume that the sequence $(a_j)_{j=1}^{\infty}$ satisfies

$$a_j = O(j^p) \qquad \text{for some } p > 1. \tag{8}$$

We now proceed to the proof of Theorem 2 and assume for the sake of contradiction that for some $0 < q < \frac{1}{4}$, there exists a ν with

$$\dim E_q > \nu > 1 - \frac{1 - 4q}{4p + 2q + \frac{1}{2}}. \tag{9}$$

The following lemma is a slight variation, due to R. C. Baker [1], of a theorem due to Frostman [7]. It enables us to formulate our problem in a way which we can look at, using methods similar to those of the previous section.

LEMMA 8. *Let $E \subset [a, b]$. For any $v < \dim E$ there exists a finite positive measure μ, supported on a compact set $C \subset E$, with* $\dim C = \dim E$, *such that if $a \leqq x < y \leqq b$, then*

$$\mu([x, y]) \leqq (y - x)^{v}. \tag{10}$$

The left hand inequality in (9) together with Lemma 8 imply the existence of a positive Borel measure μ on $[a, b]$ supported on C_q (say), a compact subset of E_q, having the same Hausdorff dimension. The idea is to show

$$D(n, x) = o(n^{-q})\mu \quad \text{a.e.}, \tag{11}$$

because this contradicts μ being supported on C_q. Showing (11) reduces, via Lemma 4, to obtaining $L^2(\mu)$ norm estimates for $nD(m, n, x)$. Firstly note that from Minkowski's inequality and Lemma 5 we have

$$\left(\int_a^b (nD(m, n, x))^2 d\mu(x)\right)^{1/2} \ll \left(\frac{n}{r} + \sum_{h=1}^{r} \frac{1}{h}\left(\int_a^b |S_h(m, n, x)|^2 d\mu(x)\right)^{1/2}\right) \tag{12}$$

for all non-negative integers $m, n \geqq 1$ and $r \geqq 1$. The next two lemmas enable us to estimate the right hand side of (12).

Together they form "the large sieve" derived by modifying a classical version due to P. X. Gallagher [9].

LEMMA 9. *For a sequence of continuously differentiable functions $(g_j(x))_{j=1}^{\infty}$ defined on $[a - \frac{1}{2}, b + \frac{1}{2}]$ and all non-negative integers $m, n \geqq 1$ and $h \geqq 1$, let*

$$s_h(x) = s_h(m, n, x) = \sum_{j=m+1}^{m+n} e(hg_j(x)).$$

Consider μ a positive Borel measure on $[a, b]$ with support having Hausdorff dimension greater than v. Suppose that if $a \leqq x < y \leqq b$ we have $\mu([x, y]) \leqq (y - x)^{v}$. Then if $\delta > 0$,

$$\int_a^b |s_h(\alpha)|^2 d\mu(\alpha)$$
$$\leqq \delta^{\nu-1} \int_{a-\delta/2}^{b+\delta/2} |s_h(x)|^2 dx + \delta^\nu \left(\int_{a-\delta/2}^{b+\delta/2} |s_h(x)|^2 dx \right)^{1/2} \left(\int_{a-\delta/2}^{b+\delta/2} |s_h'(x)|^2 dx \right)^{1/2}.$$

PROOF. For a continuously differentiable function f on $[0, 1]$ it is easily seen (by integrating by parts the second and third integrals on the right of (13)) that

$$f(x) = \int_0^1 f(u)du + \int_0^x uf'(u)du + \int_x^1 (u-1)f'(u)du. \tag{13}$$

This implies that

$$|f(\tfrac{1}{2})| \leqq \int_0^1 (|f(u)| + \tfrac{1}{2}|f'(u)|)du.$$

Hence for a real number α and a continuously differentiable function $g(t)$ defined on $[\alpha - \delta/2, \alpha + \delta/2]$ we have (after a change of variables)

$$|g(\alpha)| \leqq \int_{\alpha-\delta/2}^{\alpha+\delta/2} \left(\frac{1}{\delta} |g(t)| + \tfrac{1}{2}|g'(t)| \right) dt.$$

On setting $g(t) = s_h^2(t)$ this becomes

$$|s_h^2(\alpha)| \leqq \int_{\alpha-\delta/2}^{\alpha+\delta/2} \left(\frac{1}{\delta} |s_h^2(t)| + |s_h'(t)s_h(t)| \right) dt.$$

Integrating both sides with respect to μ this gives

$$\int_a^b |s_h^2(\alpha)| d\mu(\alpha) \leqq \int_a^b \int_{\alpha-\delta/2}^{\alpha+\delta/2} (\delta^{-1}|s_h^2(t)| + |s_h'(t)s_h(t)|)dt\, d\mu(\alpha).$$

Hence after a justified change in order or integeration we obtain

$$\int_a^b |s_h^2(\alpha)| d\mu(\alpha) \leqq \int_{a-\delta/2}^{b+\delta/2} (\delta^{-1}|s_h^2(t)| + |s_h'(t)s_h(t)|) \left(\int_{\max(a,t-\delta/2)}^{\min(b,t+\delta/2)} d\mu(\alpha) \right) dt.$$

Finally, using the fact that $\mu([x, y]) \leqq (y-x)^\nu$ and applying Cauchy's inequality to $|s_h'(t)s_h(t)|$ we have

$$\int_a^b |s_h^2(\alpha)| d\mu(\alpha)$$
$$\leqq \delta^{\nu-1} \int_{a-\delta/2}^{b+\delta/2} |s_h^2(t)| dt + \delta^\nu \left(\int_{a-\delta/2}^{b+\delta/2} |s_h^2(t)| dt \right)^{1/2} \left(\int_{a-\delta/2}^{b+\delta/2} |s_h'(t)|^2 dt \right)^{1/2}$$

as required. □

Essentially Lemma 9, in the special case where $g_j(x) = a_j x$ $(j = 1, 2, \ldots)$, is stated in [11] where it is ascribed to E. Bombieri. See [2] for a proof, however. The next lemma converts the previous one into a bound explicit in m, n and h when we choose $g_j(x) = a_j \cos a_j x$ $(j = 1, 2, \ldots)$.

LEMMA 10. *Let μ denote a positive Borel measure on $[a, b]$ with support having Hausdorff dimension greater than v and such that if $a \leqq x < y \leqq b$ then $\mu([x, y]) \leqq (y - x)^v$. Suppose for all non-negative integers m, $n \geqq 1$ and $h \geqq 1$ that* (as in Lemma 7)

$$S_h(x) = S_h(m, n, x) = \sum_{j=m+1}^{m+n} e(ha_j \cos a_j x).$$

Then there exists a constant $K_4 = K_4(a, b) > 0$ such that

$$\left(\int_a^b |S_h(m, n, x)|^2 d\mu(x) \right)^{1/2} \leqq K_4((m+n)^{p(1-v)} n^{(7-v)/8} h^{(1-v)/2}).$$

PROOF.

$$\int_{a-\delta/2}^{b+\delta/2} |S_h'(x)|^2 dx$$

$$\leqq \sum_{j=m+1}^{m+n} \sum_{k=m+1}^{m+n} h^2 a_j^2 a_k^2 \int_{a-\delta/2}^{b+\delta/2} \sin a_j x \sin a_k x e(h(a_j \cos a_j x - a_k \cos a_k x)) dx.$$

Remembering (8) and estimating the integrand on the right trivially we have, for bounded δ (by 1 say),

$$\int_{a-\delta/2}^{b+\delta/2} |S_h'(x)|^2 dx \ll h^2 (m+n)^{4p} n^2.$$

We also have from Lemma 7

$$\int_{a-\delta/2}^{b+\delta/2} |S_h(x)|^2 dx \ll n^{3/2}.$$

Lemma 9 now gives, on choosing $g_j(x) = a_j \cos a_j x$ $(j = 1, 2, \ldots)$,

$$\int_a^b |S_h(x)|^2 d\mu(x) \ll (\delta^{v-1} n^{3/2} + \delta^v (h n^{7/4} (m+n)^{2p})).$$

Letting $\delta = \frac{1}{2} n^{-1/4} (m+n)^{-2p} h^{-1}$, the lemma is established. □

We are now ready to finish off the *proof of Theorem 2*. By (9)

$$v > 1 - \frac{(1-4q)}{4p + 2q + \frac{1}{2}}. \tag{14}$$

This is equivalent to

$$q < \frac{1 - p(1-\nu) - \left(\frac{7-\nu}{8}\right)}{\left(\frac{3-\nu}{2}\right)}. \tag{15}$$

Hence we can write

$$\omega = \frac{1 - p(1-\nu) - \left(\frac{7-\nu}{8}\right)}{\left(\frac{3-\nu}{2}\right)} = q + \rho \tag{16}$$

where $\rho > 0$.

Further write

$$r = [n^{\omega}],$$

so that, as $\omega > 0$, it follows that r tends to infinity as n does. Now observe that Lemmas 5 and 10 combine to give

$$\left(\int_a^b (nD(m,n,x))^2 d\mu(x)\right)^{1/2} \ll \left(n^{1-\omega} + \sum_{h=1}^{[n^{\omega}]} \frac{1}{h}(h^{(1-\nu)/2} n^{(7-\nu)/8} (m+n)^{p(1-\nu)})\right). \tag{17}$$

Hence if we note that

$$\sum_{h=1}^{[n^{\omega}]} \frac{1}{h}(h^{(1-\nu)/2}) \ll n^{\omega(1-\nu)/2},$$

we have from (17)

$$\left(\int_a^b (nD(m,n,x))^2 d\mu(x)\right)^{1/2} \ll (n^{1-\omega} + (n+m)^{p(1-\nu)} n^{(7-\nu)/8 + \omega(1-\nu)/2}).$$

From (16)

$$1 - \omega = p(1-\nu) + \left(\frac{7-\nu}{8}\right) + \omega\left(\frac{1-\nu}{2}\right), \tag{18}$$

hence

$$\int_a^b (nD(m,n,x))^2 d\mu(x) \ll n^\theta (m+n)^\phi$$

where

$$\phi = 2p(1-\nu)$$

and

$$\theta = 1 + \left(\frac{3-\nu}{4}\right) + (1-\nu)\left\{\frac{1 - p(1-\nu) + \left(\frac{7-\nu}{8}\right)}{\left(\frac{3-\nu}{2}\right)}\right\}.$$

Now from (16) and (18) $\theta + \phi = 2 - 2\omega$ so, as $\rho > 0$ by (16), we have for all sufficiently small $\varepsilon > 0$

$$D(n,x) = O(n^{-\omega+\varepsilon}) = o(n^{-q})\mu \quad \text{a.e.},$$

as required. □

4. Exceptional sets to the expected distribution in a class of non-intervals

In this section we assume throughout that there exist positive constants K_5 and K_6 such that for the sequence $(a_j)_{j=1}^\infty$

$$K_5 j^p \leqq |a_j| \leqq K_6 j^p \qquad \text{for some } p \geqq 1. \tag{19}$$

Remember we are interested in showing that if

$$F(B,n,x) = \sum_{j=1}^n \chi_B(\langle a_j \cos s_j x\rangle) - n|B|$$

and

$$E(B) = \left\{x \in [0, 2\pi) : \overline{\lim_{n\to\infty}} \frac{1}{n}|F(B,n,x)| > 0\right\}$$

then $\dim E(B) \leqq \gamma = \max(\gamma_1, \gamma_2)$ where γ_1 and γ_2 are simple roots of polynomials $f_1(t)$ and $f_2(t)$ respectively defined by (4) and (5). Remember also that

$$f_1\left(\frac{2}{b}\right) < 0 \quad \text{and} \quad f_2\left(1 - \frac{1}{4p+\frac{1}{2}}\right) < 0. \tag{20}$$

We argue by contradiction assuming that

$$\eta = \dim E(B) > \gamma. \tag{21}$$

This means

$$f_1(\eta) > 0 \quad \text{and} \quad f_2(\eta) > 0.$$

By (20), $b\eta - 2$ and hence $b\eta - 1$ are positive. This means we can find d such that

$$(22) \qquad \frac{1 - p(1-\eta) - \left(\frac{7-\eta}{8}\right)}{\left(\frac{3-\eta}{2}\right)} > d > \max\left(\frac{2p(1-\eta)}{b\eta - 1}, \frac{p(2-3\eta)}{b\eta - 2}\right).$$

Further $2p(1-t)(bt-1)^{-1}$ and $p(2-3t)(bt-2)^{-1}$ are decreasing in t, so as $\eta > 1 - 1/(4p + \frac{1}{2})$ by (20), we know

$$(23) \qquad \max\left(\frac{2p(1-\eta)}{b\eta - 1}, \frac{p(2-3\eta)}{b\eta - 1}\right) < \frac{2p}{b(4p - \frac{1}{2}) - (4p + \frac{1}{2})} < \frac{1}{4},$$

the final inequality of (23) following because $b > 3\frac{4}{7}$ by (3) and $p \geqq 1$ by (19). This means that we can assume

$$(24) \qquad 0 < d < \tfrac{1}{4}.$$

If we now write

$$(25) \qquad s(n) = \bigcup_{i \leqq n^d} I_d \quad \text{and} \quad t(n) = \bigcup_{i > n^d} I_i$$

we have

$$F(B, n, x) = F(s(n), n, x) + F(t(n), n, x).$$

Let $(m_j)_{j=1}^{\infty}$ be the sequence $m_k = [e^{k^{1/2}}]$. The only properties of $(m_j)_{j=1}^{\infty}$ that interest us are the fact that

$$(26) \qquad \lim_{k \to \infty} \frac{m_{k+1}}{m_k} = 1,$$

and that for any $\varepsilon > 0$

$$(27) \qquad \sum_{k=1}^{\infty} m_k^{-\varepsilon} < \infty.$$

LEMMA 11. *For any set $B \subset [0, 1)$ of positive measure*

$$\left\{x : \overline{\lim_{n \to \infty}} \frac{1}{n} |F(B, n, x)| > 0\right\} \subset \left\{x : \overline{\lim_{k \to \infty}} \frac{1}{m_k} |F(B, m_k, x)| > 0\right\}.$$

Lemma 11, which hinges on (26) and is proved in [1], together with (25) implies that

(28) $$E(B) \subset P \cup Q,$$

where

$$P = \left\{ x : \overline{\lim_{k \to \infty}} \frac{1}{m_k} \left| F(s(m_k), m_k, x) \right| > 0 \right\}$$

and

$$Q = \left\{ x : \overline{\lim_{k \to \infty}} \frac{1}{m_k} \left| F(t(m_k), m_k, x) \right| > 0 \right\}.$$

We now estimate $\dim P$:

$$F(s(n), n, x) = \sum_{i \leqq n^d} (\#\{ j : 1 \leqq j \leqq n : \langle a_j \cos a_j x \rangle \in I_i \} - n \left| I_i \right|),$$

so

$$\frac{1}{n} \left| F(s(n), n, x) \right| \leqq n^d D(n, x).$$

Hence, from Theorem 2, after noting (24) we have

(29) $$\dim P \leqq 1 - \frac{1 - 4d}{4p + \frac{1}{2} + 2d}.$$

This means that $\dim Q \geqq \eta$ because the left hand side of (22) can be re-written as

$$\eta > 1 - \frac{1 - 4d}{4p + \frac{1}{2} + 2d}.$$

Now select c such that

(30) $$\eta^{-1} < \frac{c}{2} < \frac{b}{2}$$

and $\sigma < \eta$ such that

$$d > \max\left\{ \frac{2p(1-\sigma)}{c\sigma - 1}, \frac{p(2 - 3\sigma)}{(c\sigma - 2)} \right\}.$$

This is clearly possible as the right hand inequality of (22) is sharp. Since $\dim Q > \sigma$, Lemma 8 tells us there exists a positive finite Borel measure μ on

$[0, 2\pi)$, supported on a compact subset of Q, which has the same Hausdorff dimension as Q and is such that if $0 \leqq x < y \leqq 2\pi$, we have

$$\mu([x, y]) \leqq (y - x)^{\sigma}. \tag{31}$$

From (25)

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{n} \, |F(t(n), n, x)| \, d\mu(x) \\ &\leqq \int_0^{2\pi} \frac{1}{n} \, |\#\{j : 1 \leqq j \leqq n : \langle a_j \cos a_j x \rangle \in t(n)\}| \, d\mu(x) \\ &\quad + \int_0^{2\pi} |t(n)| \, d\mu(x). \end{aligned} \tag{32}$$

By (3) and (30)

$$|t(n)| = \sum_{i > n^d} |I_i| \ll \sum_{i > n^d} i^{-c},$$

which, for small enough $\varepsilon > 0$, means

$$\int_0^{2\pi} |t(n)| \, d\mu(x) \ll n^{-\varepsilon}. \tag{33}$$

We need a similar estimate for the first integral in (32). Now

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{n} \, |\#\{j : 1 \leqq j \leqq n : \langle a_j \cos a_j x \rangle \in t(n)\}| \, d\mu(x) \\ &= \frac{1}{n} \sum_{j=1}^{n} \sum_{i > n^d} \mu(E_{i,j}), \end{aligned}$$

where

$$E_{i,j} = \{x : \langle a_j \cos a_j x \rangle \in I_i\}.$$

We thus need an estimate for $\mu(E_{i,j})$.

Lemma 12. *Given an integer a other than zero and any interval $I \subset [0, 1)$ we set*

$$F = \{x \in [0, 2\pi) : \langle a \cos ax \rangle \in I\}.$$

Then $F = \bigcup_n J_n$ where the J_n are a finite number of disjoint intervals. Further, if $0 \leqq \sigma \leqq 1$ then there exists an absolute positive constant K_7 such that

$$\sum_n |J_n|^\sigma \leqq K_7(|I|^{\sigma/2}a^{(1-3\sigma/2)} + |I|^\sigma a^{2-2\sigma}).$$

PROOF. We can suppose without loss of generality that a is positive. Suppose for $m = 1, 2, \ldots, 2a$ that $A(m, x)$ are the functions alternatively monotonically decreasing or increasing (depending on m being odd or even, respectively) obtained by restricting $A(x) = a \cos ax$ to $[(m-1)\pi/a, m\pi/a)$. Note that if we assume $I = [u, u + |I|)$ and $0 \leqq u \leqq 1 - |I|$ (as we may do, again without loss of generality) then

$$F = \bigcup_{m=1}^{2a} \bigcup_{r=-a}^{a-1} \{x : A(m, x) \in [u + r, u + r + |I|)\}. \tag{34}$$

From now on, for brevity, let

$$F_{m,v} = \{x : A(m, x) \in [v, v + |I|)\}. \tag{35}$$

The $F_{m,u+r}$ $(m = 1, 2, \ldots, 2a)$ are intervals because, for each m, $A(m, x)$ is continuous and monotone in its interval of definition. We have thus expressed F as finitely many disjoint intervals. For $0 \leqq r \leqq a - 1$, by the mean value theorem there exists y in $F_{1,r+u}$ and z in $F_{1,r+1-|I|}$ such that

$$|F_{1,r+u}|C(y) = |I| = |F_{1,r+1-|I|}|C(z),$$

where $C(y)$ refers to the modulus of the derivative of $A(x)$ evaluated at y. Now $C(x)$ is monotonically increasing on $[0, \pi/2a)$ so assuming, as we may without loss of generality, that $F_{1,r+u}$ and $F_{1,r+1-|I|}$ are disjoint we have for all $0 \leqq r \leqq a - 1$

$$|F_{1,r+u}| \leqq |F_{1,r+1-|I|}|. \tag{36}$$

Similarly $C(x)$ is monotonically decreasing on $[\pi/2a, \pi/a)$ so, for $-a \leqq r \leqq -1$,

$$|F_{1,r+u}| \leqq |F_{1,r}|. \tag{37}$$

Now considering the symmetries of the graph of $A(x)$ we have, fixing r and u for all m, that

$$|F_{m,u+r}| = |F_{1,u+r}| \tag{38}$$

and, for $1 \leqq r \leqq a$,

$$|F_{m,r-|I|}| = |F_{m,-r}|. \tag{39}$$

Hence combining (34), (35), (36), (37), (38) and (39) we have, for $F = \bigcup_n J_n$,

$$\sum_n |J_n|^\sigma \leqq 4a \sum_{r=1}^{a} |F_1, r - |I||^\sigma. \tag{40}$$

Now we know that $\sin x \geqq 2x/\pi$ on $[0, \pi/2)$. Hence integrating we obtain $\cos x \leqq 1 - x^2/\pi$ on $[0, \pi/2)$. Rescaling this,

$$a \cos ax \leqq a - \frac{a^3 x^2}{\pi}$$

on $[0, \pi/2a)$. The mean value theorem now tells us that

$$|F_{1,a-|I|}|^\sigma \leqq \left| \left\{ x : \left(a - \frac{a^3 x^2}{\pi} \right) \in [a - |I|, a) \right\} \right|^\sigma = \pi^{\sigma/2} |I|^{\sigma/2} a^{-3\sigma/2}. \tag{41}$$

Further $A''(x) = -a^2 A(x)$, hence $A(x)$ is concave when positive. Remember that if $h(y)$ is concave in the interval $[x, z]$ and $y \in [x, z]$ then

$$(h(x) - h(z))(z - y) \leqq (h(y) - h(z))(z - x).$$

In consequence we have, if we choose $h(t) = A(1, t)$, $x = A^{-1}(1, r + 1 - |I|)$, $y = A^{-1}(1, r)$ and $z = A^{-1}(1, r - |I|)$ for $1 \leqq r \leqq a - 1$,

$$\begin{aligned} |F_1, r - |I||^\sigma &= |A^{-1}(1, r - |I|) - A^{-1}(1, r)|^\sigma \\ &\leqq |I|^\sigma (A^{-1}(1, r - |I|) - A^{-1}(1, r - |I| + 1))^\sigma, \end{aligned}$$

where $A^{-1}(m, x)$ is the inverse function of $A(m, x)$ in x for fixed m. Using the concavity of x^σ, for $0 \leqq \sigma \leqq 1$, we have

$$\sum_{r=1}^{a-1} (A^{-1}(1, r + 1 - |I|) - A^{-1}(1, r - |I|))^\sigma \leqq (a - 1)^{1-\sigma} \left(\frac{\pi}{2a} \right)^\sigma. \tag{42}$$

The proof of the lemma is now over because (40), (41) and (42) combine to give

$$\sum_n |J_n|^\sigma \leqq K_7 (|I|^{\sigma/2} a^{(1 - 3\sigma/2)} + |I|^\sigma a^{2 - 2\sigma})$$

as required. □

Noting (31) and (19), Lemma 12 immediately gives

$$\frac{1}{n} \sum_{j=1}^{n} \sum_{i > n^d} \mu(E_{i,j}) \ll \frac{1}{n} \sum_{j=1}^{n} \sum_{i > n^d} (j^{p(1 - 3\sigma/2)} i^{-c\sigma/2} + i^{-c\sigma} j^{2p(1-\sigma)})$$

which is

(43) $$\ll (n^{2p(1-\sigma)-d(c\sigma-1)} + n^{p(1-3\sigma/2)-d(c\sigma/2-1)}) \ll n^{-\varepsilon},$$

for some $\varepsilon > 0$. Thus we know by (27), (33) and (43) that

$$\sum_{k=1}^{\infty} \int_0^{2\pi} \frac{1}{m_k} |F(t(m_k), m_k, x)| \, d\mu(x) < \infty.$$

This means

$$\sum_{k=1}^{\infty} \frac{1}{m_k} |F(t(m_k), m_k, x)| < \infty \qquad \mu\text{-a.e.}$$

In particular

$$F(t(m_k), m_k, x) = o(m_k) \qquad \mu\text{-a.e.}$$

This contradicts the fact that μ is supported on a compact subset of Q with the same Hausdorff dimension and so we have proved that $\dim E(B) \leqq \gamma$ as required. □

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